ON THE MOTION OF GYROSCOPIC DEVICES UNDER THE INFLUENCE OF RANDOM FORCES

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The motion of gyroscopic devices under ship's rolling conditions was usually considered assuming the roll sinusoidal. The works of Sveshnikov [1] showed that the rolling motion of a ship can be considered a stationary random process, and obtained the correlation function and spectral density for this process. The motion of some ship devices under these conditions was studied by Rivkin [2].

The present paper considers the motion of a powered gyrostabilizer, plane gyropendulum and a gyrocompass subject to irregular ship rolling; determines the dispersion of a stabilization angle for the powered gyrostabilizer; and derives an expression for intercardinal deviation of the gyrocompass.

1. Powered gyrostabilizer in the presence of irregular ship rolling. The equations of motion for the powered gyrostabilizer subject to ship rolling are of the following form [3]:

$$A\alpha'' - H\beta' - Ki_2 = j(j-1)I\theta'' - \zeta(\alpha' - \theta') \tag{1.1}$$

$$B\beta'' + H\alpha' = 0, \quad l_1 i_1' + r_1 i_1 - s\beta = 0, \quad l_2 i_2' + r_2 i_2 + c\alpha' - \varkappa i_1 = c\theta'$$

where

$$A = A_1 + j^2 I, \qquad K = jk_1\Phi, \qquad c = jk_2\Phi \tag{1.2}$$

Here α is the angle rotation of the gyrostabilizer frame about its axis, β the angle of rotation of the gyroscope about the axis of its housing, θ the angle of rolling, A_1 the moment of inertia of the gyrostabilizer frame together with the stabilizer object and the gyroscope about the axis of the frame, B the equatorial moment of inertia of the gyroscope, A the angular momentum of the gyroscope, i_1 the current in the amplifier circuit; l_1/r_1 the time constant of this circuit, i_2 the armature current of the unloading motor, l_2/r_2 the time constant of the armature circuit, I the armature moment of inertia of the unloading motor, j the gear ratio of the transmission from the shaft of the unloading motor to the axis of the gyrostabilizer frame (the number of axes of the gear transmission is assumed to be odd), ζ the coefficient of viscous friction at the supports of the gyrostabilizer frame and Φ the magnetic flux created by the excitation coil of the unloading motor with independent excitation).

We shall limit ourselves to the consideration of gyrostabilizers whose time constants l_1/r_1 and l_2/r_2 in the control circuits are so small that their influence can be neglected.

In this case one can let $l_1 = l_2 = 0$ in the equations of motion (1.1) which then will assume the form

$$\alpha'' + \frac{n}{A}\alpha' - \frac{H}{A}\beta' - \frac{m}{A}\beta = \frac{a}{A}\theta'' + \frac{n}{A}\theta', \quad \beta'' + \frac{H}{B}\alpha' = 0 \qquad (1.3)$$

where

$$m = \frac{s\varkappa K}{r_1 r_2}, \qquad n = \zeta + \frac{Kc}{r_2}, \qquad a = j(j-1)I \qquad (1.4)$$

Let D be the operator for time differentiation ($D = d/dt$), and introducing matrices

$$f(D) = \begin{Vmatrix} D^2 + \frac{n}{A}D & -\left(\frac{H}{A}D + \frac{m}{A}\right) \\ \frac{H}{B}D & D^2 \end{Vmatrix}, \qquad e(D) = \begin{Vmatrix} \frac{a}{A}D^2 + \frac{n}{A}D \\ 0 \end{Vmatrix}$$

$$y = \begin{Vmatrix} \alpha \\ \beta \end{Vmatrix} \qquad (1.5)$$

we shall alter the system of equations (1.3) by the matrix equation

$$f(D)\,y = e(D)\,\theta(t) \qquad (1.6)$$

From Equation (1.6) it follows that

$$y = Y(D)\,\theta(t) \qquad \left(Y(D) = \frac{F(D)\,e(D)}{\Delta(D)}\right) \qquad (1.7)$$

Here $F(D)$ is the adjoint matrix for $f(D)$

$$F(D) = \begin{Vmatrix} D^2 & \frac{H}{A}D + \frac{m}{A} \\ -\frac{H}{B}D & D^2 + \frac{n}{A}D \end{Vmatrix} \qquad (1.8)$$

and $\Delta(D)$ is the determinant of matrix $f(D)$

$$\Delta(D) = D\left(D^3 + \frac{n}{A} D^2 + q^2 D + \frac{m}{H} q^2\right) \qquad \left(q^2 = \frac{H^2}{AB}\right) \tag{1.9}$$

Here q is the frequency of nutational oscillations of the gyrostabilizer.

Matrix $Y(D)$ is the matrix transfer function for the gyrostabilizer. In accordance with (1.7), (1.8) and (1.5) the matrix $Y(D)$ may be expressed in the following form:

$$Y(D) = \frac{1}{\Delta(D)} \left\| \begin{matrix} \frac{a}{A} D^4 + \frac{n}{A} D^3 \\ -\left(\frac{Ha}{AB} D^3 + \frac{Hn}{AB} D^2\right) \end{matrix} \right\| \tag{1.10}$$

The angle α of the rotation of the gyrostabilizer frame and the angle β of the rotation of the gyroscope housing, according to (1.5), (1.7) and (1.10) shall be determined by the following operational expressions:

$$\begin{aligned} \alpha &= Y_{11}(D)\theta(t) = \frac{1}{\Delta(D)}\left(\frac{a}{A} D^4 + \frac{n}{A} D^3\right)\theta(t) \\ \beta &= Y_{21}(D)\theta(t) = -\frac{1}{\Delta(D)}\left(\frac{Ha}{AB} D^3 + \frac{Hn}{AB} D^2\right)\theta(t) \end{aligned} \tag{1.11}$$

As was shown in [1], the irregular rolling motion of a ship can be considered a stationary random process, the correlation function of which has the form

$$R_1(\tau) = L_1 e^{-\mu|\tau|}\left(\cos\varepsilon\tau + \frac{\mu}{\varepsilon}\sin\varepsilon|\tau|\right) \tag{1.12}$$

where L_1 is the dispersion of the angle of roll and μ and ϵ are characteristic coefficients for a given ship. The corresponding spectral density for the correlation function (1.12) is

$$S_1(\omega) = L_1 \frac{4\mu\nu^2}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}, \qquad \nu^2 = \varepsilon^2 + \mu^2 \tag{1.13}$$

Deviations of the gyrostabilizer caused by the rolling of the ship can be evaluated by the dispersion of the stabilization angle, i.e. by the dispersion α^2 of the angle of rotation of the gyrostabilizer frame. According to (1.11) and (1.13)

$$\overline{\alpha^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y_{11}(i\omega)|^2 S_1(\omega)\, d\omega \tag{1.14}$$

where in accordance with (1.11) and (1.9)

$$|Y_{11}(i\omega)|^2 = \frac{\left(\frac{a}{A}\right)^2 \omega^6 + \left(\frac{n}{A}\right)^2 \omega^4}{\omega^6 + \left[\left(\frac{n}{A}\right)^2 - 2q^2\right]\omega^4 + \left(q^4 - 2\frac{mn}{HA}q^2\right)\omega^2 + \left(\frac{m}{H}\right)^2 q^4} \tag{1.15}$$

Expression (1.14) may be transformed to

$$\overline{\alpha^2} = 4\mu\nu^2 L_1 I_5, \qquad I_5 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)h(-i\omega)}d\omega \tag{1.16}$$

Here

$$g(i\omega) = b_0(i\omega)^8 + b_1(i\omega)^6 + b_2(i\omega)^4 + b_3(i\omega)^2 + b_4 \tag{1.17}$$

$$h(i\omega) = a_0(i\omega)^5 + a_1(i\omega)^4 + a_2(i\omega)^3 + a_3(i\omega)^2 + a_4(i\omega) + a_5$$

and the coefficients of the polynomials (1.17) will be

$$b_0 = 0, \quad b_1 = -\left(\frac{a}{A}\right)^2, \quad b_2 = \left(\frac{n}{A}\right)^2, \quad b_3 = 0, \quad b_4 = 0$$

$$a_0 = 1, \quad a_1 = 2\mu + \frac{n}{A}, \quad a_2 = 2\mu\frac{n}{A} + q^2 + \nu^2$$

$$a_3 = \left(2\mu + \frac{m}{H}\right)q^2 + \frac{n}{A}\nu^2, \quad a_4 = \left(2\mu\frac{m}{H} + \nu^2\right)q^2, \quad a_5 = \frac{m}{H}q^2\nu^2 \tag{1.18}$$

The integrals of the type (1.17) were evaluated by Phillips [4]. In the present case

$$I_5 = \frac{M_5}{2N_5} \tag{1.19}$$

where

$$M_5 = a_0 b_1(a_3a_4 - a_2a_5) + a_0b_2(a_0a_5 - a_1a_4) \tag{1.20}$$

$$N_5 = -\begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix} = [a_0^2a_5^2 - 2a_0a_1a_4a_5 - a_0a_2a_3a_5 + a_0a_3^2a_4 + a_1^2a_4^2 + a_1a_2^2a_5 - a_1a_2a_3a_4] \tag{1.21}$$

and the expression (1.16) for the dispersion of the stabilization angle will be of the form

$$\overline{\alpha^2} = \frac{2\mu\nu^2 L_1 M_5}{N_5} \tag{1.22}$$

As an example we will consider a gyrostabilizer with the following parameters:

$$A = 50 \text{ kg m sec}^2,\ B = 0.04 \text{ kg m sec}^2,\ H = 30 \text{ kg m sec}$$

$$m = 100 \text{ kg m},\ n = 250 \text{ kg m sec},\ a = 5 \text{ kg m sec}^2$$

Furthermore, as follows from (1.9), the nutation frequency of the gyrostabilizer is $q = 21.2\ \text{sec}^{-1}$; the nutation period is $T = 2\pi/q = 0.296$ sec. The roll correlation function parameters are $\mu = 0.1\ \text{sec}^{-1}$, $\nu = 0.8\ \text{sec}^{-1}$.

Dispersion of the roll angle is $L_1 = \theta^2 = 0.03$; the average mean square value of the ship roll angle is $\sqrt{L_1} = 0.173$, i.e. about $10°$.

From this data we find the dispersion of the stabilization angle $\alpha^2 = 0.60 \times 10^{-6}$, and the average mean-square values of the stabilization angle and of the gyroscope housing angle

$$\sqrt{\overline{\alpha^2}} = 0.777.10^{-3} \approx 2.7' \qquad \sqrt{\overline{\beta^2}} = 0.332 \approx 19°$$

2. Plane gyropendulum in the presence of irregular roll. The equations of motion for a plane gyropendulum subject to irregular roll are of the form

$$A\alpha'' + H\beta' + lP\alpha \pm M\beta = a\theta'' - n(\alpha' - \theta') \tag{2.1}$$

$$B\beta'' + E\beta' - H\alpha' + \varkappa\beta = 0, \qquad a = \frac{lP}{g} r \tag{2.2}$$

Here α is the angle of rotation of the gyropendulum about its axis, β the angle of rotation of the gyroscope about the axis of its housing, θ the roll angle of the ship, A and B the corresponding moments of inertia, H the moment of momentum of the gyroscope, lP the static moment of the pendulum, M the magnitude of the radial correcting moment, κ the stiffness of the spring connecting the gyroscope housing with the outer Cardan ring, E and n the coefficients of viscous friction and r the distance from the ship's center of roll to the axis of the gyropendulum suspension.

Limiting ourselves to the study of the precessional motion of the gyropendulum, we eliminate in the equations of motion (2.1) the inertia terms $A\alpha''$ and $B\beta''$. Then Equations (2.1) become

$$\beta' + \frac{n}{H}\alpha' + \frac{lP}{H}\alpha + \frac{M}{H}\beta = \frac{a}{H}\theta'' + \frac{n}{H}\theta',$$

$$-\alpha' + \frac{E}{H}\beta' + \frac{\varkappa}{H}\beta = 0 \tag{2.3}$$

Let D be the operator for time differentiation ($D = d/dt$) and introducing the matrices

$$f(D)=\left\|\begin{matrix}\frac{n}{H}D+\frac{lP}{H} & D+\frac{M}{H}\\ -D & \frac{E}{H}D+\frac{\varkappa}{H}\end{matrix}\right\|,\qquad y=\left\|\begin{matrix}\alpha\\ \beta\end{matrix}\right\|,\qquad e(D)=\left\|\begin{matrix}\frac{a}{H}D^2+\frac{n}{H}D\\ 0\end{matrix}\right\| \tag{2.4}$$

we replace the system of equations (2.3) by the matrix equation

$$f(D)\,y=e(D)\,\theta(t) \tag{2.5}$$

From Equation (2.5) it follows that

$$y=Y(D)\,\theta(t)\qquad\left(Y(D)=\frac{F(D)\,e(D)}{\Delta(D)}\right) \tag{2.6}$$

Here $F(D)$ is the adjoint matrix for $f(d)$

$$F(D)=\left\|\begin{matrix}\frac{E}{H}D+\frac{\varkappa}{H} & -\left(D+\frac{M}{H}\right)\\ D & \frac{n}{H}D+\frac{lP}{H}\end{matrix}\right\| \tag{2.7}$$

and $\Delta(D)$ is the determinant of matrix $f(D)$

$$\Delta(D)=(1+\sigma)D^2+\zeta D+k^2 \tag{2.8}$$

where

$$\sigma=\frac{nE}{H^2},\qquad \zeta=\frac{\varkappa n+lPE}{H^2}+\frac{M}{H},\qquad k^2=\frac{\varkappa lP}{H^2} \tag{2.9}$$

Matrix $Y(D)$ is the matrix transfer function for the system. In accordance with (2.6), (2.7) and (2.8) it may be expressed as

$$Y(D)=\frac{1}{\Delta(D)}\left\|\begin{matrix}\frac{aE}{H^2}D^3+\left(\sigma+\frac{a\varkappa}{H^2}\right)D^2+\frac{\varkappa n}{H^2}D\\ \frac{a}{H}D^3+\frac{n}{H}D^2\end{matrix}\right\| \tag{2.10}$$

The angle of rotation α of the gyropendulum and the angle of rotation β of the gyroscope housing will be determined, in agreement with (2.6) and (2.7), by the expressions

$$\alpha=Y_{11}(D)\,\theta(t)=\frac{1}{\Delta(D)}\left[\frac{aE}{H^2}D^3+\left(\sigma+\frac{a\varkappa}{H^2}\right)D^2+\frac{\varkappa n}{H^2}D\right]\theta(t)$$

$$\beta=Y_{21}(D)\,\theta(t)=\frac{1}{\Delta(D)}\left(\frac{a}{H}D^3+\frac{n}{H}D^2\right)\theta(t) \tag{2.11}$$

The dispersion $\overline{\alpha}^2$ of the angle of rotation of the gyroscopic pendulum and $\overline{\beta}^2$ of the angle of rotation of the gyroscope housing will be determined by the following expressions:

$$\overline{\alpha^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y_{11}(i\omega)|^2S_1(\omega)\,d\omega,\qquad \overline{\beta^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y_{21}(i\omega)|^2S_1(\omega)\,d\omega \tag{2.12}$$

Here

$$|Y_{11}(i\omega)|^2 = \frac{1}{K}\left\{\left(\frac{aE}{H^2}\right)^2 \omega^6 + \left[\sigma^2 + \left(\frac{a\varkappa}{H^2}\right)^2\right]\omega^4 + \left(\frac{\varkappa n}{H^2}\right)^2 \omega^2\right\}$$

$$|Y_{21}(i\omega)|^2 = \frac{1}{K}\left[\left(\frac{a}{H}\right)^2 \omega^6 + \left(\frac{n}{H}\right)^2 \omega^4\right] \qquad (2.13)$$

$$K = (1+\sigma)^2\omega^4 + [\zeta^2 - 2(1+\sigma)k^2]\omega^2 + k^4$$

In expressions (2.12), $S_1(\omega)$ denotes the spectral density of the roll angle θ, which according to (1.13) is of the form

$$S_1(\omega) = L_1 \frac{4\mu\nu^2}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}$$

The expressions (2.13) may be transformed into

$$\overline{\alpha^2} = 4\mu\nu^2 L_1 I_4, \qquad \overline{\beta^2} = 4\mu\nu^2 L_1 J_4 \qquad (2.14)$$

Here

$$I_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)h(-i\omega)}\,d\omega, \qquad J_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(i\omega)}{h(i\omega)h(-i\omega)}\,d\omega \qquad (2.15)$$

$$g(i\omega) = b_0(i\omega)^6 + b_1(i\omega)^4 + b_2(i\omega)^2 + b_3$$

$$G(i\omega) = B_0(i\omega)^6 + B_1(i\omega)^4 + B_2(i\omega)^2 + B_3 \qquad (2.16)$$

$$h(i\omega) = a_0(i\omega)^4 + a_1(i\omega)^3 + a_2(i\omega)^2 + a_3(i\omega) + a_4$$

The coefficients of polynomials (2.16) are

$$b_0 = -\left(\frac{aE}{H^2}\right)^2, \quad b_1 = \sigma^2 + \left(\frac{a\varkappa}{H^2}\right)^2, \quad b_2 = -\left(\frac{\varkappa n}{H^2}\right)^2, \quad b_3 = 0$$

$$B_0 = -\left(\frac{a}{H}\right)^2, \quad B_1 = \left(\frac{n}{H}\right)^2, \quad B_2 = 0, \quad B_3 = 0 \qquad (2.17)$$

$$a_0 = 1+\sigma, \quad a_1 = 2\mu(1+\sigma) + \zeta, \quad a_2 = 2\mu\zeta + k^2 + \nu^2(1+\sigma)$$

$$a_3 = 2\mu k^2 + \zeta\nu^2, \qquad a_4 = k^2\nu^2$$

In the case when all zeroes of the function $h(D)$ are located on the left half-plane of the complex variable D, the integrals (2.15), according to Phillips [4], are

$$I_4 = \frac{M_1}{N}, \qquad J_4 = \frac{M_2}{N} \qquad (2.18)$$

Here

$$M_1 = b_0(a_1a_4 - a_2a_3) + a_0a_3b_1 - a_0a_1b_2, \quad M_2 = B_0(a_1a_4 - a_2a_3) + a_0a_3B_1$$

$$N = 2a_0(a_1a_2a_3 - a_0a_3^2 - a_1^2a_4) \qquad (2.19)$$

After substitution of a_i, b_i and B_i, according to (2.17) we have

$$M_1 = \left(\frac{aE}{H^2}\right)^2 [2\mu k^4 + 4\mu^2\zeta k^2 + 2\mu\zeta^2\nu^2 + \zeta\nu^4(1+\sigma)] + \qquad (2.20)$$

$$+ \left[\sigma^2 + \left(\frac{a\varkappa}{H^2}\right)^2\right](1+\sigma)(2\mu k^2 + \zeta\nu^2) + \left(\frac{\varkappa n}{H^2}\right)^2 (1+\sigma)[2\mu(1+\sigma)+\zeta]$$

$$M_2 = \left(\frac{a}{H}\right)^2 [2\mu k^4 + 4\mu^2\zeta k^2 + 2\mu\zeta^2\nu^2 + \zeta\nu^4(1+\sigma)] + \left(\frac{n}{H}\right)^2 (2\mu k^2 + \zeta\nu^2)(1+\sigma)$$

$$N = 4\mu\zeta(1+\sigma)\{k^4 + 2[(2\mu^2 - \nu^2)(1+\sigma) + \mu\zeta]k^2 + $$
$$+ \nu^4(1+\sigma)^2 + 2\mu\zeta\nu^2(1+\sigma) + \zeta^2\nu^2\}$$

As an example we will consider a plane gyropendulum, the parameters of which are

$$\frac{lP}{H} = \frac{\varkappa}{H} = 0.02\,\text{sec}^{-1}, \qquad \frac{M}{H} = 0.01\,\text{sec}^{-1}, \qquad E = n = 0$$

The distance from the center of the ship's roll to the support axis of the gyropendulum is r = 3 m. Then according to (2.9)

$$\frac{a}{H} = 0.006\,\text{sec}, \qquad \sigma = 0, \qquad \zeta = 0{,}01\,\text{sec}^{-1}, \qquad k = 0.02\,\text{sec}^{-1}$$

The period of the gyropendulum is $T = 2\pi/k$ = 314 sec. The ship's roll correlation function parameters are

$$\mu = 0.1\,\text{sec}^{-1},\ \nu = 0.8\,\text{sec}^{-1}$$

Dispersion of the ship's roll angle $L_1 = \theta^2 = 0.03$; the average mean-square value $\sqrt{L_1} = 0.173$, i.e. about 10°.

According to (2.16) for the given data the dispersion of α and β will be as follows:

$$\overline{\alpha^2} = 4.37\cdot 10^{-10}, \qquad \overline{\beta^2} = 0.69\cdot 10^{-6}$$

The average mean-square value of the gyropendulum rotational angle α is

$$\sqrt{\overline{\alpha^2}} = 2.09\cdot 10^{-5}$$

i.e. about 4.3 seconds of arc. The average mean-square value of the gyroscope housing rotational angle β is

$$\sqrt{\overline{\beta^2}} = 0.83\cdot 10^{-3}$$

i.e. about 3 minutes of arc.

Thus, during ship's roll, oscillations with considerable amplitudes occur about the axis of the gyroscope housing. The amplitude of oscillations about the gyropendulum support axis is small, since the average mean-square value of the angle α is extremely small.

This result confirms the advantage of a twin gyroscopic vertical, containing two plane gyropendulums as compared with a single-rotor gyropendulum under the conditions of ship's roll.

3. Intercardinal deviation of a gyrocompass. The equations of motion for a double-rotor gyrocompass in the presence of ship's roll can be expressed in the following form:

$$\begin{gathered}A_1\alpha'' + H\beta' + HU\cos\varphi\,\alpha = H\frac{v_N}{R} + \frac{lP}{g}W_2\gamma \\ A_2\beta'' - H\alpha' + lP\beta + lP(1-\rho)\vartheta = HU\sin\varphi - \frac{lP}{g}W_2 \\ \vartheta' + F\vartheta + F\beta = -F\frac{W_2}{g},\quad A_3\gamma'' + K\delta' + lP\gamma = \frac{lP}{g}W_1 \\ A_4\delta'' + m\delta' - K\gamma' + \varkappa\delta = KU\cos\varphi\end{gathered}\tag{3.1}$$

where

$$H = 2B\cos\varepsilon,\qquad K = 2B\sin\varepsilon\tag{3.2}$$

Here α is the gyrocompass rotational angle in azimuth, β and γ the angles of elevation, respectively, for the North and West gyrosphere diameters above the horizontal plane, δ the angle of precession for the gyroscopes with respect to the gyrosphere, ϑ the inclination of the fluid level in the hydraulic damper with respect to the equatorial plane of the gyrosphere, A_1, A_2, A_3, A_4, the corresponding moments of inertia, B the eigen moment of each gyroscope, 2ϵ the angle between the rotor axes of the gyroscopes, lP the static moment of the gyrosphere, κ the stiffness of the spring connecting the gyro housing with the gyrosphere, U the angular velocity of the Earth's diurnal rotation, ϕ the latitude of observation; v_N the Northern component of ship's velocity, W_1 and W_2 the Eastern and Western components of translational acceleration for the gyro suspension point and R is the radius of the Earth.

For the case of rectilinear uniform motion of the ship in the presence of roll we may set

$$W_1 \approx -r\theta''\cos\psi,\qquad W_2 \approx r\theta''\sin\psi\tag{3.3}$$

where θ is the roll angle, ψ the ship's course and r distance from the support point of the gyroscope to the straight line passing through the center of the ship's roll and directed parallel to the longitudinal axis of the ship.

Studying only the precessional motion of the gyrocompass we eliminate in the equations of motion (3.1) the inertia terms $A_1\alpha''$, $A_2\beta''$, $A_3\gamma''$ and $A_4\delta''$. Then Equations (3.1) become

$$\beta' + U\cos\varphi\,(\alpha - \alpha^*) = \frac{k^2}{U\cos\varphi}\,\frac{r\sin\psi}{g}\,\theta''\gamma$$

$$\alpha' - \frac{k^2}{U\cos\varphi}(\beta - \beta^*) - \frac{k^2(1-\rho)}{U\cos\varphi}(\vartheta - \vartheta^*) = \frac{k^2}{U\cos\varphi}\,\frac{r\sin\psi}{g}\,\theta''$$

$$\vartheta' + F(\vartheta - \vartheta^*) + F(\beta - \beta^*) = -F\,\frac{r\sin\psi}{g}\,\theta'' \tag{3.4}$$

$$\gamma'' + \zeta\gamma' + n^2\gamma = -\frac{r\cos\psi}{g}(\zeta\theta''' + n^2\theta'')$$

where

$$k^2 = \frac{lPU\cos\varphi}{H}, \qquad n^2 = \frac{\varkappa lP}{K^2}, \qquad \zeta = \frac{m}{\varkappa}\,n^2 \tag{3.5}$$

$$\alpha^* = \frac{v_N}{RU\cos\varphi}, \qquad \beta^* = \frac{HU\sin\varphi}{\rho lP}, \quad \vartheta^* = -\frac{HU\sin\varphi}{\rho lP} \tag{3.6}$$

Assuming, in accordance with (1.12), that θ is a stationary random process we will define, for the generalized coordinates of the gyrocompass, their mathematical expectancies, which we denote as

$$x_1 = M[\alpha - \alpha^*], \qquad x_2 = M[\beta - \beta^*], \qquad x_3 = M[\vartheta - \vartheta^*] \tag{3.7}$$

The value of these quantities for $t \to \infty$ we denote by x_1^*.

For the coordinate γ, as can be seen from the last equation of (3.4), the steady value of the mathematical expectancy is

$$x_4^* = 0 \tag{3.8}$$

The quantities x_1, x_2, x_3 satisfy the following equations:

$$x_2' + U\cos\varphi\, x_1 = E$$

$$x_1' - \frac{k^2}{U\cos\varphi}\,x_2 - \frac{k^2(1-\rho)}{U\cos\varphi}\,x_3 = 0, \qquad x_3' + Fx_2 + Fx_3 = 0 \tag{3.9}$$

where

$$E = \frac{k^2}{U\cos\varphi}\,\frac{r\sin\psi}{g}\,R_c(0) \tag{3.10}$$

and $R_c(\tau)$ denotes the cross-correlation function for the random processes θ'' and γ.

The quantity $R_c(0)$ may be computed as follows. The spectral density S_1 for the roll angle θ and the spectral density S_2 for the random process θ'' are, according to (1.13)

$$S_1(\omega) = \frac{4\mu\nu^2 L_1}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}, \qquad S_2(\omega) = \frac{4\mu\nu^2 L_1\omega^4}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2} \tag{3.11}$$

And since according to the fourth equation of (3.4)

$$\gamma = -\frac{r\cos\psi}{g}\,\frac{\zeta D + n^2}{D^2 + \zeta D + n^2}\,\theta'' \qquad \left(D = \frac{d}{dt}\right) \tag{3.12}$$

the cross-spectral density $S_c(\omega)$ for the random processes θ'' and γ may be expressed in the form

$$S_c(\omega) = -4\mu\nu^2 L_1 \frac{r\cos\psi}{g}\,\frac{(i\zeta\omega + n^2)\,\omega^4}{(-\omega^2 + i\zeta\omega + n^2)\,[(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2]} \tag{3.13}$$

Noting that

$$R_c(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_c(\omega)\,e^{i\omega\tau}\,d\omega \tag{3.14}$$

one can transform the expression (3.10) into

$$E = -2\mu\nu^2 L_1 \frac{k^2}{U\cos\varphi}\,\frac{r^2\sin 2\psi}{g^2}\,I \tag{3.15}$$

where

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-(n^2 - \zeta^2)\,\omega^6 + n^4\omega^4 - i\zeta\omega^7}{[(\omega^2 - n^2)^2 + \zeta^2\omega^2]\,[(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2]}\,d\omega \tag{3.16}$$

Since the subintegral function is odd

$$\int_{-\infty}^{\infty} \frac{\omega^7}{[(\omega^2 - n^2)^2 + \zeta^2\omega^2]\,[(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2]}\,d\omega = 0$$

Therefore, the expression (3.16) becomes

$$I = I_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)\,h(-i\omega)}\,d\omega \tag{3.17}$$

Here

$$g(i\omega) = b_0(i\omega)^6 + b_1(i\omega)^4 + b_2(i\omega)^2 + b_3 \tag{3.18}$$

$$h(i\omega) = a_0(i\omega)^4 + a_1(i\omega)^3 + a_2(i\omega)^2 + a_3(i\omega) + a_4$$

$$b_0 = n^2 - \zeta^2, \qquad b_1 = n^4, \qquad b_2 = 0, \qquad b_3 = 0 \tag{3.19}$$

$$a_0 = 1, \quad a_1 = 2\mu + \zeta, \quad a_2 = 2\mu\zeta + n^2 + \nu^2, \quad a_3 = 2\mu n^2 + \zeta\nu^2, \quad a_4 = \nu^2 n^2$$

For the integral (3.17), according to Phillips [4], we will have

$$I_4 = \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1}{2a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)} \tag{3.20}$$

Upon substitution of a_i and b_i from (3.19) this expression becomes

$$I_4 = \frac{(\nu^2 + 2\mu\zeta - 4\mu^2)n^4 - (\nu^4 + 2\mu\zeta\nu^2 - 4\mu^2\zeta^2)n^2 + (\nu^2 + 2\mu\zeta)\zeta^2\nu^2}{4\mu[n^4 + 2(2\mu^2 + \mu\zeta - \nu^2)n^2 + \nu^2(\nu^2 + 2\mu\zeta + \zeta^2)]} \tag{3.21}$$

The system of homogeneous differential equations resulting from (3.9) for $E = 0$ has the characteristic equation

$$D^3 + FD^2 + k^2D + \rho k^2F = 0 \tag{3.22}$$

For $\rho < 1$, which is always true in gyrocompasses, all roots of the characteristic equation (3.22) will lie on the left half-plane of the complex variable D and the integrals of the above-mentioned system of homogeneous equations will asymptotically tend to zero for $t \to \infty$. Therefore, during ship's roll, a process of sufficiently long duration, the quantities x_1, x_2, x_3 will attain their steady-state values which in agreement with (3.9) will be as follows:

$$x_1^* = \frac{E}{U\cos\varphi}, \qquad x_2^* = 0, \qquad x_3^* = 0 \tag{3.23}$$

In accordance with (3.15), the expression for x_1^* becomes

$$x_1^* = aI_4 \sin 2\psi \qquad \left(a = -2\mu\nu^2 L_1\left(\frac{k}{U\cos\varphi}\right)^2 \frac{r^2}{g^2}\right) \tag{3.24}$$

where I_4 is defined above by the expression (3.21).

The quantity x_1^* is the intercardinal deviation of the gyrocompass. It vanishes for cardinal courses of the ship ($\psi = 0°$, $90°$, $180°$, $270°$) and attains its largest values for intercardinal courses ($\psi = 45°$, $135°$, $225°$, $315°$).

As an example we will determine the intercardinal deviation for a gyrocompass with $k = 1.24 \times 10^{-3}$ sec^{-1} which corresponds to the natural period in azimuth $T_k = 84.4$ min. The latitude ϕ of the point of observation is assumed equal to 60°, so that $U\cos\phi = 3.646 \times 10^{-5}$ sec^{-1}. The distance from the gyrocompass support point to the straight line passing through the ship's center of roll is $r = 2$ m. The roll correlation function parameters are $\mu = 0.1\ \sec^{-1}$, $\nu = 0.8\ \sec^{-1}$. The dispersion of the roll angle is $L_1 = 0.03$, to which corresponds the average mean-square value of the roll angle

$$\sqrt{L_1} = 0.173$$

i.e. about 10°.

The intercardinal deviation of the gyrocompass, in agreement with (3.24), is then determined by the expression

$$x_1^* = -0.186\, I_4 \sin 2\psi$$

where I_4, in agreement with (3.21), depends upon the natural frequency n of the gyrocompass in y-direction. The values of the function $I_4(T_n)$, where $T_n = 2\pi/n$, are given in the table.

TABLE

T_n, сек	$I_4\ (\zeta = 0)$	$I_4\ (\zeta = 0.2\,n)$	T_n, сек	$I_4\ (\zeta = 0)$	$I_4\ (\zeta = 0.2\,n)$
0.2	1.501	4.642	8.7	—3.613	—1.309
0.5	1.505	2.762	8.8	—3.540	—1.371
1.0	1.521	2.149	8.9	—3.445	—1.419
2.0	1.589	1.899	9.0	—3.337	—1.455
3.0	1.715	1.905	9.5	—2.749	—1.500
4.0	1.921	2.015	10	—2.245	—1.420
5.0	2.247	2.178	11	—1.565	—1.170
6.0	2.698	2.455	12	—1.160	—0.944
6.6	2.837	1.953	15	—0.599	—0.537
7.0	2.491	1.498	20	—0.291	—0.272
7.1	2.278	1.344	30	—0.118	—0.112
7.2	1.992	1.174	40	—0.064	—0.061
7.3	1.624	0.990	50	—0.040	—0.039
7.4	1.169	0.795	100	$-0.994\cdot10^{-2}$	$-0.953\cdot10^{-2}$
7.5	0.633	0.591	200	$-0.272\cdot10^{-2}$	$-0.238\cdot10^{-2}$
7.6	0.029	0.382	300	$-0.110\cdot10^{-2}$	$-0.105\cdot10^{-2}$
7.7	—0.615	0.178	400	$-0.617\cdot10^{-3}$	$-0.592\cdot10^{-3}$
7.8	—1.263	—0.033	500	$-0.395\cdot10^{-3}$	$-0.379\cdot10^{-3}$
7.9	—1.875	—0.233	600	$-0.274\cdot10^{-3}$	$-0.263\cdot10^{-3}$
8.0	—2.416	—0.422	700	$-0.201\cdot10^{-3}$	$-0.193\cdot10^{-3}$
8.1	—2.860	—0.599	800	$-0.154\cdot10^{-3}$	$-0.148\cdot10^{-3}$
8.2	—3.202	—0.759	900	$-0.122\cdot10^{-3}$	$-0.117\cdot10^{-3}$
8.3	—3.438	—0.904	1000	$-0.987\cdot10^{-4}$	$-0.948\cdot10^{-4}$
8.4	—3.582	—1.031	1100	$-0.816\cdot10^{-4}$	$-0.783\cdot10^{-4}$
8.5	—3.648	—1.140	1200	$-0.685\cdot10^{-4}$	$-0.658\cdot10^{-4}$
8.6	—3.653	—1.233			

As can be seen from the table, the intercardinal deviation is sufficiently small for gyrocompasses with the period T_n in y direction comparatively large, of the order of 15-20 min, as is the case for double-rotor gyrocompasses. Thus, in the given example, for the case when

T_n = 20 min, and sin $2\psi = 1$, the intercardinal deviation consists of only 0.04 minutes of arc.

The intercardinal deviation is also small in the neighborhood of the zero of the function $I_4(n)$, located near the point $n = \nu$. Thus, for $\zeta = 0$, the function I_4 becomes zero for

$$n = \nu/\sqrt{1 - 4\mu^2/\nu^2}$$

The relationship among the parameters for which the condition $n \approx \nu$ is satisfied occurs in gyrocompasses with mercury ballistic vessels [5].

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